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Asymptotic Theory of Electro[illegible]
in an Inhomogeneous Anisotropic Pl[illegible]

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Abstract

Recently developed "ray methods" for the asymptotic solution of linear partial differential equations are applied to a system of time-dependent magneto-ionic equations. These are equations for E, H, and the electron current, J. For time-harmonic waves J can be eliminated and the equations reduce to the usual equations of the magneto-ionic theory. The time-dependent equations contain a large parameter λ, the plasma frequency, which serves as an expansion parameter. In the "strongly anisotropic" case both the gyro frequency and plasma frequency are large. In the simpler "weakly anisotropic" case only the plasma frequency is large. Both cases can be treated by the general method presented. The weakly anisotropic case is analyzed in greater detail. One result of this analysis is a formula for the rate of rotation of the electric vector about a curved ray in an inhomogeneous medium. The formula reduces to that of the well-known Faraday rotation for the special case of a plane wave propagating in a homogeneous medium.

1. Introduction

In recent years asymptotic methods have been developed for the solution of a large class of problems for linear partial differential equations. These problems involve a parameter and the methods provide one or more terms of the asymptotic expansion, say for large values of the parameter, of the solution of a given initial or boundary value problem. They are often applicable to problems for which no exact solution method is known, and even for problems that can be solved exactly it frequently happens that only the asymptotic expansion of the solution is sufficiently simple to be useful in practical applications. An important class of asymptotic methods is characterised by the fact that certain curves, or "rays", play a central role in the theory. The rays are important because the functions which make up the various terms of the expansion satisfy ordinary differential equations along these curves. Often these equations can be solved explicitly to yield relatively simple formulas that yield considerable insight into the physical nature of the solutions. These "ray methods" are closely related to and generalize the methods of geometrical optics and Keller's geometrical theory of diffraction [5].

In this paper we apply ray methods to the solution of a system of partial differential equations that describe electromagnetic wave propagation in the ionosphere. By combining Maxwell's equations, in their time-dependent form, with a simple equation of motion for free electrons we obtain a hyperbolic system of partial differential equations for the electric and magnetic field $\underline{E}$ and $\underline{H}$, and the electron current $\underline{J}$.

In section 2 we show that for time-harmonic waves we can eliminate **J** and this system reduces to the usual equations of the magnetoionic theory [1,2]. Therefore we refer to the equations for **E**, **H**, and **J** as "time-dependent magnetoionic equations". They contain a large parameter, λ, the average plasma frequency, which serves as our expansion parameter. An equivalent dimensionless parameter is $\lambda\, \ell/c$, where ℓ is a characteristic length of the problem and c is the speed of light. In section 2 we distinguish two cases, both of which can be treated by the general method discussed in section 3. In the "strongly anisotropic" case both the gyro frequency and plasma frequency are large parameters. In the "weakly anisotropic" case only the plasma frequency is large.

The discussion in section 3 is a brief but self-contained treatment of the method developed in [4] and [6] for the asymptotic solution of dispersive symmetric hyperbolic systems. It includes an improved treatment of the solution of the "transport equations" for the case of a double root. This case is important for the application considered here.

In section 4 we apply the general results of section 3 to the time-dependent magnetoionic equations. For simplicity we consider the weakly anisotropic case and find that there are three types of modes. The "propagating modes" are the most important. They contain only "wave frequencies" greater than the plasma frequency, propagate with "group velocity" less than c, and are attenuated with "decay exponent" proportional to the collision frequency. In addition to the propagating modes there are "standing electric modes" corresponding to an oscillation with the local plasma frequency, and static magnetic modes. For both of these modes the group velocity is zero.

In section 5 we consider the propagating modes and study the direction of the field vectors. We find that they are perpendicular to the ray but rotate about it. We derive an explicit formula for the rate of rotation in an inhomogeneous medium, for which the rays may be curved. For the special case of a time-harmonic plane wave propagating in a homogeneous medium the formula reduces to that of the well-known Faraday rotation.

Although we have pursued the weakly anisotropic case further, we emphasize that the strongly anisotropic case is also included in the general theory of section 3. Further analysis of this case requires the study of the dispersion relation and the null vectors of the "dispersion matrix". The difficulties that arise here are algebraic and essentially the same as those that arise in the usual magneto-ionic theory, where they have been analyzed in detail [1,2]. It seems clear that many of the results of the standard theory can be applied to our asymptotic theory, but we have not undertaken to do that here.

In M.K.S. units Maxwell's equations for free space take the form

$$\nabla \times H - \epsilon \frac{\partial E}{\partial t} = j, \tag{1}$$

$$\nabla \times E + \mu \frac{\partial H}{\partial t} = 0, \tag{2}$$

$$\nabla \cdot (\mu H) = 0, \tag{3}$$

$$\nabla \cdot (\epsilon E) = \rho. \tag{4}$$

Here E and H are the electric and magnetic field vectors and ϵ and μ are the dielectric constant and magnetic permeability of free space. It follows immediately from (2) that $\frac{\partial}{\partial t}\nabla \cdot (\mu H) = 0$, hence (3) is automatically satisfied if it is satisfied at a given time, say $t=0$. Thus (3) may be interpreted as a condition on the initial data of an initial value problem for the electromagnetic field. We shall view (4) as a formula for the charge density ρ. Then, except for the initial data condition (3) we need only consider (1) and (2).

In order to describe electromagnetic wave propagation in a plasma we assume that the current density vector j consists of two parts,

$$j = j_1 + J. \tag{5}$$

Here $j_1(x,t)$ is the [illegible] external current, and J is the current due to the motion of free electrons. (We neglect the effect of heavy ions.) Thus

$$J = eNv, \tag{6}$$

where e is the electronic charge, N is the number density of electrons, and v is their average velocity. We assume that v satisfies the equation of

motion (Newton's equation)

$$m\underline{v}_t = e(\underline{E} + \underline{v}\times\mu\underline{H}_0) - m\nu\underline{v} . \tag{7}$$

Here m is the electronic mass and $\underline{H}_0$ is a (prescribed) external magnetic field. We assume that $|\underline{H}| << |\underline{H}_0|$ and we have neglected the contribution of $\underline{H}$ to the Lorentz force, i.e. to the bracketed term in (7). The second force term in (7) is a damping force due to collisions, and ν is called the collision frequency. Thus (6) and (7) yield

$$\underline{J}_t = \frac{e^2N}{m}\underline{E} - \frac{e\mu}{m}\underline{H}_0\times\underline{J} - \nu\underline{J} . \tag{8}$$

We now introduce the plasma frequency ϕ defined by

$$\epsilon\phi^2 = N\frac{e^2}{m} , \tag{9}$$

and the gyro frequency γ defined by

$$\gamma = \frac{|e|\mu}{m}|\underline{H}_0| , \quad (|e| = -e) . \tag{10}$$

Then

$$\frac{-e\mu}{m}\underline{H}_0 = \gamma\underline{H} , \tag{11}$$

where $\underline{H}$ is a unit vector. Thus (1), (2) and (8) become

$$\epsilon\underline{E}_t - \nabla\times\underline{H} = -\underline{J}_1 , \tag{12}$$

$$\mu\underline{H}_t + \nabla\times\underline{E} = 0 , \tag{13}$$

$$\underline{J}_t - \gamma\underline{H}\times\underline{J} + \nu\underline{J} - \epsilon\phi^2\underline{E} = 0 . \tag{14}$$

These partial differential equations for $\underline{E}$, $\underline{H}$, and $\underline{J}$ will be called the time-dependent magneto-ionic equations because they reduce to the usual equations of the magneto-ionic theory in the special case of a time-harmonic wave. In order to see this we assume that $\underline{E}$, $\underline{H}$, $\underline{J}$ and $\underline{J}_1$ are functions of x

[illegible]

with [illegible] [illegible]

$$[illegible] \qquad (15)$$

$$[illegible] \qquad (16)$$

$$[illegible] \qquad (17)$$

If we obtain [illegible] for which [illegible]

$$[illegible] \qquad (18)$$

where

$$M = \begin{bmatrix} [illegible] & 0 & 0 \\ 0 & \gamma & [illegible] \\ 0 & [illegible] & \gamma \end{bmatrix}, \quad [illegible] \qquad (19)$$

and

$$\gamma = \frac{[illegible]}{[illegible]} . \qquad (20)$$

Now (17) yields

$$[illegible] \qquad (21)$$

hence [illegible]

$$[illegible] \, M^{-1} \qquad (22)$$

Thus, by eliminating J from (15)-(17) we obtain the equation

$$[illegible] \qquad (23)$$

$$[illegible] = 0 , \qquad (24)$$

which are the usual equations of the magneto-ionic theory [1,2]. It is interesting to note that whereas (21) involves the relatively simple matrix M, in order to eliminate J we must introduce the much more complicated matrix M^{-1}.

[illegible]
equations can be conveniently written as a single matrix equation. For any vector $\underline{z} = (z_1, z_2, z_3)$ we introduce the 3×3 matrix

$$(\underline{z}) = \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} \tag{25}$$

Then for any vector $\underline{A}$ it is easily seen that

$$(\underline{z})\underline{A} = \underline{z}\times\underline{A} \tag{26}$$

We also introduce the vector $\underline{I}$ defined by

$$\underline{J} = \sqrt{\varepsilon\phi}\,\underline{I}. \tag{27}$$

Then (14) becomes

$$\underline{I}_t - \gamma\underline{U}\times\underline{I} + \nu\underline{I} - \sqrt{\varepsilon\phi}\,\underline{E} = 0, \tag{28}$$

and the system of equations (12), (13), (28) can be written in the block matrix notation

$$\begin{bmatrix} \varepsilon\frac{\partial}{\partial t} & -(\nabla) & \sqrt{\varepsilon\phi} \\ (\nabla) & \mu\frac{\partial}{\partial t} & 0 \\ -\sqrt{\varepsilon\phi} & 0 & \frac{\partial}{\partial t}+\nu-\gamma(\underline{U}) \end{bmatrix} \begin{bmatrix} \underline{E} \\ \underline{H} \\ \underline{I} \end{bmatrix} = \begin{bmatrix} -\underline{J}_1 \\ 0 \\ 0 \end{bmatrix}. \tag{29}$$

Each entry in the square matrix on the left side of (29) represents a 3×3 matrix. Thus, e.g. $\sqrt{\varepsilon\phi} = \sqrt{\varepsilon\phi}\, I_3$ where I_3 is the 3×3 identity matrix. Furthermore each entry in the column matrices represents a column vector of dimension 3.

We now introduce three 9×9 matrices A^1, A^2, A^3 with entries either zero or one. These matrices can be defined simultaneously by writing

[illegible]

[illegible] in regard to cross and we will study it further in sections 4 and 5.

We note that in both cases (15) and (37) [illegible] is anti-symmetric. [illegible]

[illegible] (25) [illegible]
[illegible] that the (dissipative) collision frequency ν as a large parameter.

[illegible] theory of hyperbolic systems

Let (1) be considered hyperbolic system of the form

$$A^0 u_t + \sum_{\nu=1}^{n} A^\nu u_{x_\nu} + \lambda B u = f, \tag{1}$$

where $A^0, A^1, \dots, A^n, B, f$ are smooth [illegible] functions of t and $x = (x_1, \dots, x_n)$, $u(t,x)$ and $f(t,x)$ are [illegible] column vectors, and λ is a large positive parameter. Here we present a brief but self-contained summary of the method of [5,6] for the special case of an asymptotically conservative symmetric hyperbolic equation, i.e. an equation of the form (1) for which A^0 is positive definite, $A^1, \dots, A^n$ are Hermitian, and B is anti-Hermitian. These conditions are all satisfied by the system introduced in section 2.

We consider a formal asymptotic expansion

$$[illegible] \sum_{j=0}^{\infty} [illegible] \tag{2}$$

which is to be a solution of (1) with $f=0$. The function s is called the phase function and the functions u_j are called amplitude coefficients. We introduce the partial derivatives

$$\omega = -\partial s/\partial t \quad ; \; k_\nu = \frac{\partial s}{\partial x_\nu} \quad (\nu = 1, \dots, n) \tag{3}$$

and the dispersion matrix

$$G(t, x, \omega, k) = \sum_{\nu=1}^{n} k_\nu A^\nu - iB - \omega A^0. \tag{4}$$

By inserting (3) into (1) with $\beta = 0$, [illegible] of each power of λ, we obtain the recurrence [illegible]

$$[illegible] \quad (5)$$

and

$$[illegible] \quad (6)$$

Nontrivial solutions are [illegible] the dispersion relation,

$$\det G = \det\left(\sum_{\nu=1}^{n} [illegible]\right) = 0, \quad (7)$$

is satisfied. Let

$$\omega = h(t, x, k) \quad (8)$$

be a root of (7) of multiplicity q. This means that [illegible] q linearly independent null vectors r such that

$$G r_\mu = 0, \qquad \mu = 1, \ldots, q. \quad (9)$$

[illegible], for each [illegible] the matrix $\sum_{\nu=1}$ [illegible] positive definite, there exist [illegible] multiplicities q is equal to [illegible] linearly independent. The vectors r_μ can be so chosen that

$$(r_\mu, A^0 r_{\mu'}) = \delta_{\mu\mu'} \qquad \mu, \mu' = 1, \ldots, q. \quad (10)$$

Equation (8), considered as a first order partial differential equation for $s(t, x)$ [see (2)] can be solved by the method of characteristics [3]. If we introduce the group velocity vector g with components

$$g_\nu = \partial h / \partial k_\nu, \qquad \nu = 1, \ldots, n, \quad (11)$$

*If a and b are column vectors the inner product (a, b) is defined by $(a, b) = \sum_{j=1} a_j^* b_j$. The star denotes the complex conjugate. For matrices it denotes the Hermitian conjugate.

the n - parameters. For each t the ray transformation $y \to x(t,y)$ defines a mapping from y-space to x-space with jacobian

$$j(t) = j(t,y) = \det\left(\frac{\partial x_i}{\partial y_j}\right) = \sum_{j=1}^{n} \frac{\partial x_i}{\partial y_j} \operatorname{cof} \frac{\partial x_i}{\partial y_j} , \tag{25}$$

Here we have introduced the expansion of the determinant by cofactors of the i^{th} row. Since a determinant vanishes when two rows are the same, we have the identity,

$$\sum_{j=1}^{n} \frac{\partial x_k}{\partial y_j} \operatorname{cof} \frac{\partial x_i}{\partial y_j} = j\, \delta_{ik} . \tag{26}$$

By differentiating (25) and using (26) and (12) we obtain

$$\frac{1}{j}\frac{dj}{dt} = \frac{1}{j}\sum_{i,j} \frac{\partial^2 x_i}{\partial y_j \partial t} \operatorname{cof} \frac{\partial x_i}{\partial y_j} = \frac{1}{j}\sum_{i,j,k} \frac{\partial}{\partial x_k}\left(\frac{\partial x_i}{\partial t}\right)\left[\frac{\partial x_k}{\partial y_j} \operatorname{cof} \frac{\partial x_i}{\partial y_j}\right]$$

$$= \sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial x_i}{\partial t}\right) = \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} = \operatorname{div} g . \tag{27}$$

This identity which relates the jacobian of the ray transformation and the divergence of the group velocity vector will be used shortly.

We now impose the following condition on the coefficient matrices of (1):

$$[r_\beta, (C + C^*) r_\alpha] - \sum_{\nu=0}^{n} (r_\beta, A^{\nu}_{x_\nu} r_\alpha) = \kappa \delta_{\beta\alpha} \; ; \; \beta,\alpha = 1,\ldots,q. \tag{28}$$

The condition states that the q×q matrix whose entries are given by the left side of (28) reduces to a scalar. The condition is trivially satisfied if q=1. For the magneto-ionic equations of section 2, the matrices A^{ν} are constant hence (28) reduces to a condition on the matrix C, which we shall examine in section 4. Now from (22), (17), (27) and (28) we obtain

[illegible]

The energy density is an [illegible] random function that characterizes the lasing amplitude [illegible] In some cases we can obtain further information [illegible] by integrating the transport equations (34). The most important cases are those for which the multiplicity q is one or two.

[illegible] (2[illegible]) and (3[illegible]) [illegible]

$$\frac{d\sigma}{dt} + [illegible] = 0 \quad [illegible]$$

where

$$[illegible] (\xi, G\xi) . \quad [illegible]$$

[illegible] introduc[illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

$$[illegible] + [illegible] \frac{d\theta}{dt} + |\sigma| [illegible] = 0. \quad [illegible]$$

The imaginary part of (35) yields [illegible]

$$\theta(t) = \theta(t_0) - \int_{t_0}^{t} [illegible] \quad [illegible]$$

since $|\sigma| = [illegible]$, [illegible] yields [illegible]. [illegible]

[illegible] multiplied by [illegible] [illegible] (35), [illegible]

$$[illegible] \left[\frac{[illegible](t_0)}{[illegible](t)}\right]^{1/2} [illegible] \left\{ - \int_{t_0}^{t} [illegible] \right\} . \quad (37)$$

[illegible] (2[illegible]) we see that

$$[illegible] = [illegible] \quad (38)$$

Since [illegible] (37) yields

$$[illegible](t) = \left[\frac{[illegible]}{[illegible](t)}\right]^{1/2} \exp\left\{ - \int_{t_0}^{t} [\tfrac{1}{2} [illegible] + [illegible]] \right\} [illegible] \quad (39)$$

[illegible] it is convenient to write (34) in matrix notation. We introduce the [illegible] matrix [illegible] and the column matrix [illegible] with entries [illegible]. Then (34) becomes

$$[illegible] = 0 \quad (40)$$

[illegible] (3[illegible]) and [illegible] yield

$$[illegible] \frac{1}{3} \frac{d[illegible]}{dt} , [illegible] \quad (41)$$

We now set

$$\underline{a} = e^{iU}\,\underline{\beta}\,. \tag{42}$$

Then (40) yields

$$\dot{\underline{\beta}} + P\,\underline{\beta} = 0, \tag{43}$$

where

$$P = T + \frac{\dot{U}}{\hbar}\,, \tag{44}$$

and it follows from (41) that

$$P + P^{\dagger} = 0. \tag{45}$$

Then P is anti-hermitian.

We now assume that the elements of the matrix T are all real. Thus we impose the condition

$$\tau^{*}_{\beta\alpha} = \tau_{\beta\alpha} \qquad\qquad \beta,\alpha = 1,2. \tag{46}$$

Then P is real and antisymmetric, i.e.

$$P = \begin{bmatrix} 0 & p \\ -p & 0 \end{bmatrix}. \tag{47}$$

It follows from (44) that

$$p = \tau_{12} = -\tau_{21}\,. \tag{48}$$

We now set

$$U(t) = \begin{bmatrix} \cos\ \delta(t) & -\sin\ \delta(t) \\ \sin\ \delta(t) & \cos\ \delta(t) \end{bmatrix} \tag{49}$$

Then by computing $\dot{U}$ and PU from (47) and (49) it is easy to see that the solution of (43) is

$$\underline{\beta}(t) = U(t)\,\underline{\beta}(t_0) \tag{50}$$

[illegible] the usual electro-magnetic energy density, and the last term is the [illegible] energy per unit volume of the electrons. For any vector $\underline{k}$, we see from (1.30) that

$$\underline{k}\cdot\underline{S} = \frac{1}{2}\sum_{\nu=1}^{3} k_\nu (u, A^\nu u) - \frac{1}{2}\left[\underline{E}\cdot\underline{k}\times\underline{H} - \underline{E}\cdot\underline{k}\times\underline{H}\right] = \underline{k}\cdot\underline{E}\times\underline{H} \,. \tag{56}$$

Thus,

$$\underline{S} = \underline{E}\times\underline{H}, \tag{57}$$

i.e. $\underline{S}$ reduces to the usual electromagnetic Poynting vector.

For complex solutions we define S_0 and $\underline{S}$ by

$$S_\nu = \frac{1}{2}(\mathrm{Re}\,u, A^\nu \mathrm{Re}\,u) \,, \quad \nu=0,1,\ldots,n; \tag{58}$$

and for our asymptotic solution $u \sim z_0 \exp(i\lambda s)$ we find that

$$S_\nu \sim \frac{1}{8}\left[(z_0, A^\nu z_0^*) + (z_0^*, A^\nu z_0) + e^{2i\lambda s}(z_0, A^\nu z_0) + e^{-2i\lambda s}(z_0^*, A^\nu z_0^*)\right]. \tag{59}$$

We note that the last two terms contain oscillatory factors, hence if we average (59) over a short time interval, the last two terms will be asymptotically zero for $\lambda \to \infty$. Thus the average value of S_ν is given by

$$\langle S_\nu \rangle \sim \frac{1}{8}\left[(z_0, A^\nu z_0^*) + (z_0^*, A^\nu z_0)\right] \,. \tag{60}$$

If A^ν is real (60) reduces to

$$\langle S_\nu \rangle \sim \frac{1}{4}(z_0, A^\nu z_0), \tag{61}$$

and we see from (30) that

$$w \sim \langle S_0 \rangle . \tag{62}$$

That is why we have called w the average energy density. Furthermore we see from (19), (17), and (30) that

$$\langle S_\nu \rangle \sim w q_\nu \,, \tag{63}$$

hence the average energy density, [illegible] Poynting vector, and the [illegible] [illegible] meaningful equation

[illegible] (64)

4. [illegible]

The equations discussed in section [illegible] of the [illegible] treated in section [illegible], therefore the theory developed there applies to them. In order to obtain more specific results it is necessary to discuss the null eigenvectors r_j of the matrix G and the roots $\omega = h(\xi, x)$ of the dispersion relation. This can be done easily in the weakly anisotropic case introduced in section 2 and therefore we restrict our attention here to that case.

From (2.30), (2.31), and (2.37) we see that

$$[\text{illegible}] = \begin{bmatrix} [\text{illegible}] & [\text{illegible}] & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] & 0 \\ i\sqrt{s}\,b & 0 & -\omega \end{bmatrix} \tag{1}$$

If we introduce the nine-dimensional column vector r with components (P, Q, R), then $Gr = 0$ if and only if

$$\omega P + k \times Q + i\sqrt{s}\, b R = 0, \tag{2}$$

$$k \times P - \mu\omega Q = 0, \tag{3}$$

$$i\sqrt{s}\, b P - \omega R = 0 \; . \tag{4}$$

[illegible] we obtain

$$[\text{illegible}] \tag{5}$$

From [illegible] it follows that

$$(\omega^2 - b^2)\, k \cdot P = 0. \tag{6}$$

These equations have non-trivial solutions in three mutually exclusive cases:

$$\text{case I: } \omega^2 \neq b^2, \omega \neq 0 \; ; \quad \text{case II: } \omega = \pm b; \text{ case III: } \omega = 0. \tag{7}$$

We consider first the most important case

Case I: Propagating modes.

Here it follows from (7), (6), and (5) that

$$\underset{\sim}{k}\cdot\underset{\sim}{P} = 0 \tag{8}$$

and

$$\omega^2 - b^2 - c^2k^2 = 0 . \tag{9}$$

Therefore there are two roots of the dispersion relation

$$\omega = h(\underset{\sim}{k},\underset{\sim}{x}) = \pm\sqrt{c^2k^2 + b^2} . \tag{10}$$

We shall see that each is of multiplicity 2. For each root (10) we choose real linearly independent vectors $\underset{\sim}{P}_j$ $(j=1,2)$ which satisfy (8) and determine corresponding vectors $\underset{\sim}{Q}_j$, and $\underset{\sim}{R}_j$ from (3) and (4). Then we impose the normalization condition (3.10) and find that $2\varepsilon\underset{\sim}{P}_i\cdot\underset{\sim}{P}_j = \delta_{ij}$. Therefore we must set $\underset{\sim}{P}_j = (2\varepsilon)^{-1/2}\underset{\sim}{e}_j$ where $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ are real unit vectors which are mutually orthogonal and orthogonal to $\underset{\sim}{k}$. Then the components of $\underset{\sim}{r}_j$ are

$$\underset{\sim}{r}_j = (\underset{\sim}{P}_j,\underset{\sim}{Q}_j,\underset{\sim}{R}_j) = \left(\frac{1}{\sqrt{2\varepsilon}}\underset{\sim}{e}_j,\ \frac{1}{\sqrt{2\varepsilon}\mu\omega}\underset{\sim}{k}\times\underset{\sim}{e}_j,\ \frac{ib}{\sqrt{2}\,\omega}\underset{\sim}{e}_j\right),\quad j = 1,2, \tag{11}$$

We introduce a unit vector $\underset{\sim}{e}_3$ in the ray direction. Then the group velocity vector is given by

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{g} = \nabla_{\underset{\sim}{k}} h = \frac{c^2}{\omega}\underset{\sim}{k} = \frac{1}{\varepsilon\mu\omega}\underset{\sim}{k}, \tag{12}$$

and (11) becomes

$$\underset{\sim}{r}_1 = \left(\frac{1}{\sqrt{2\varepsilon}}\underset{\sim}{e}_1,\ \sqrt{\frac{\varepsilon}{2}}\,g\underset{\sim}{e}_2,\ \frac{ib}{\sqrt{2}\,\omega}\underset{\sim}{e}_1\right), \tag{13}$$

$$\underset{\sim}{r}_2 = \left(\frac{1}{\sqrt{2\varepsilon}}\underset{\sim}{e}_2,\ -\sqrt{\frac{\varepsilon}{2}}\,g\underset{\sim}{e}_1,\ \frac{ib}{\sqrt{2}\,\omega}\underset{\sim}{e}_2\right). \tag{14}$$

Since two linearly independent null vectors g_1, g_2 [illegible]
(10), each is of multiplicity 2.

The matrix C is given by (2.37). Since g_1 and g_2 are real except for the factor i in the last three components, it is easily seen from (3.22) that $r_{\alpha\beta}$ is real. Thus condition (3.42) is satisfied. Furthermore,

$$Q = c^{-4} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{15}$$

and it follows from (13) and (14) that condition (3.20) is satisfied with η given by

$$\eta = \frac{[illegible]}{[illegible]} . \tag{16}$$

Thus we may use (3.58) and (3.57) to determine g_0. The properties of this solution formula for g_0 will be discussed further in section 5.

Let us finally note that (3.14) becomes

$$\frac{ds}{dt} + [illegible] \tag{17}$$

and s can be found by integrating this equation along a ray. The dispersion relation (9) for propagating modes is identical to the dispersion relation for the equation studied in [7]. Therefore solutions constructed from these modes will have many properties in common with the solutions studied in [7]. In particular we note that the group speed

$$g = \frac{c^2 |k|}{|\omega|} = c\sqrt{1-b^2/\omega^2} = c\sqrt{1-\frac{b^2}{\omega^2}} , \tag{18}$$

[illegible]

... multipoles ... corresponding to ... [illegible] ... propagating ... [illegible]

[illegible]

... the ... field vectors. Since ... and the ... field ... [illegible] ... these modes ... we refer to these modes as ... magnetic modes.

Thus we find that there are four propagating modes, ... admissible ... magnetic modes. ... in all, which is the ... number for ... [illegible] ... admissible ... can be excited with any given ... This is determined by the ...

The [illegible] and [illegible] the rate vector [illegible] which

[illegible], and p_0 is the

[illegible] in the [illegible], [illegible] of the ray.

[illegible] from the coefficients [illegible] (5.21)

[illegible]

and

$$[illegible] \qquad (11)$$

[illegible] we may say that the [illegible] "rotated" [illegible] direction determined by [illegible], and from [illegible] we see that the direction rotates relative to the [illegible] p_0 [illegible] rigid body [illegible] at an angular speed given by [illegible]. It is not different [illegible] for an infinite homogeneous [illegible] plane wave propagating parallel to the external magnetic field, [illegible] the exact formula for the well-known Faraday rotation [illegible], using our formula (10) for the generalized Faraday rotation, and [illegible] we can determine the leading amplitude coefficient a_0 by integration along a ray.

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